Parallel Sorting Algorithms

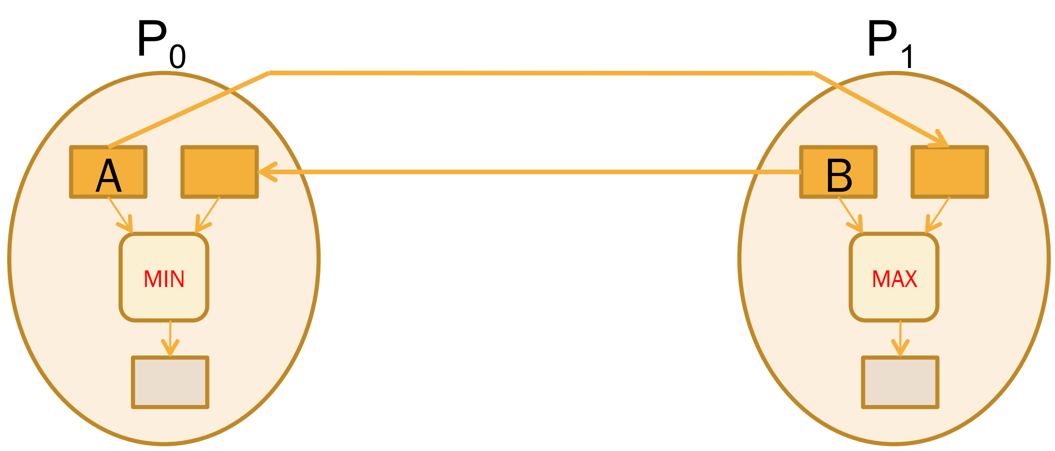
Sorting Algorithms Review

- Bubble Sort: $O(n^2)$
- Insertion Sort: $O(n^2)$
- Quick Sort: $O(n \log n)$
- Heap Sort: $O(n \log n)$
- Merge Sort: $O(n \log n)$

- The best we can expect from a sequential sorting algorithm using p processors (if distributed evenly among the n elements to be sorted) is $O(n \log n) / p \sim O(\log n)$.

Compare and Exchange Sorting Algorithms

- Form the basis of several, if not most, classical sequential sorting algorithms.
- Two numbers, say A and B, are compared between P_0 and P_1.



Bubble Sort

- Generic example of a “bad” sorting algorithm.

- Algorithm:
  - Compare neighboring elements.
  - Swap if neighbor is out of order.
  - Two nested loops.
  - Stop when a whole pass completes without any swaps.

- Performance:
  - Worst: $O(n^2)$
  - Average: $O(n^2)$
  - Best: $O(n)$

	0	1	2	3	4	5
start:	1	3	8	0	6	5
after pass 1:	1	3	0	6	5	8
after pass 2:	1	0	3	5	6	8
after pass 3:	0	1	3	5	6	8
after pass 4:	0	1	3	5	6	8

fin.

"The bubble sort seems to have nothing to recommend it, except a catchy name and the fact that it leads to some interesting theoretical problems."

- Donald Knuth, The Art of Computer Programming

Odd-Even Transposition Sort (also Brick Sort)

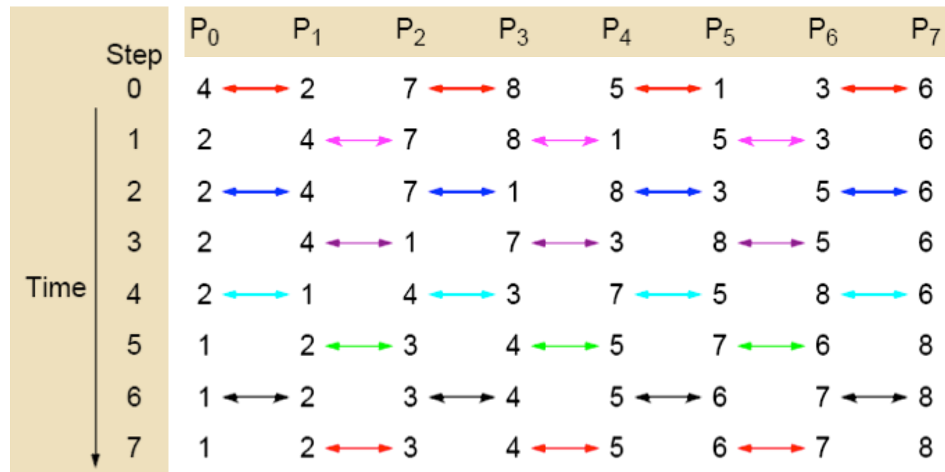
- Simple sorting algorithm that was introduced in 1972 by Nico Habermann who originally developed it for parallel architectures (“Parallel Neighbor-Sort”).

- A comparison sorting algorithm that is related to bubble sort because it shares a similar approach.

- It compares all (odd-even) indexed pairs of adjacent elements in a list and switches them if they are out of order. The next step repeats this process for (even-odd) indexed pairs and continues alternating until the list is sorted.

- The odd-even transposition sort makes use of a pipelining technique to ultimately run many phases of the bubble sort in parallel.

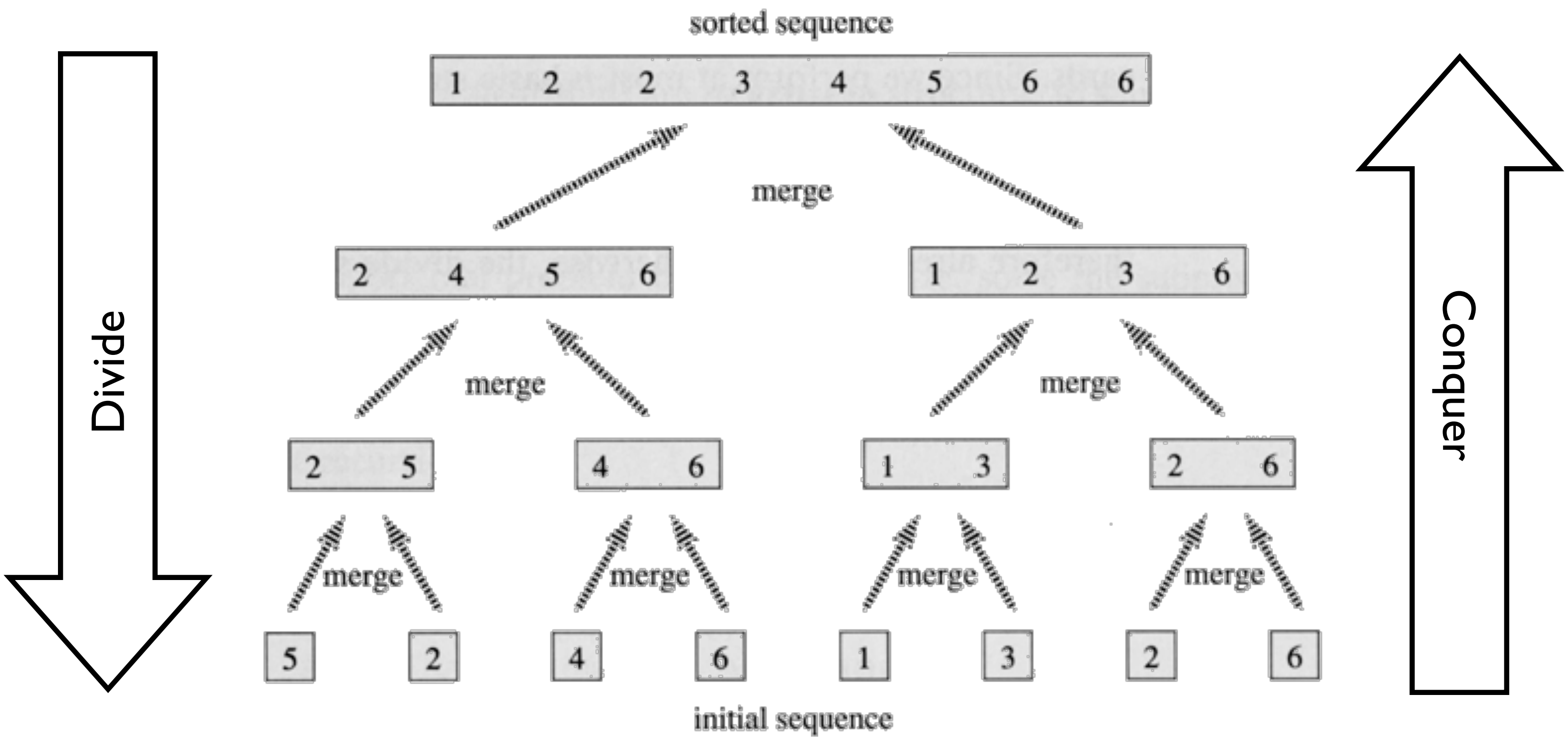
- The running time of this algorithm is $O(n^2)/p \sim O(n)$



MergeSort

- Divide and conquer approach
  - Characterized by dividing the problem into sub-problems of same form as larger problem. Further divisions into still smaller sub-problems, usually done by recursion.
  - Recursive divide-and-conquer amenable to parallelization because separate processes can be used for divided parts. Also usually data is naturally localized.
- Divide the n values to be sorted into two halves
- Recursively sort each half using MergeSort
  - Base case n=1 no sorting required
- Merge the two halves (fundamental operation)
  - O(n) operation

MergeSort


Divide
sorted sequence
1 2 2 3 4 5 6 6
merge
2 4 5 6
1 2 3 6
merge
merge
2 5
4 6
1 3
2 6
merge
merge
merge
merge
5
2
4
6
1
3
2
6
initial sequence
Conquer

Merge Operation

0	1	6	0 < 5	5	6	8

0					

0	1	6	1 < 5	5	6	8

0	1				

0	1	6	6 > 5	5	7	8

0	1	5			

0	1	6	6 < 7	5	7	8

0	1	5	6		

Now, do rest of second array..

0	1	6	6 < 7	5	7	8

0	1	5	6	7	8

fin.

O(n) running time because each element is considered (n-1 comparisons)

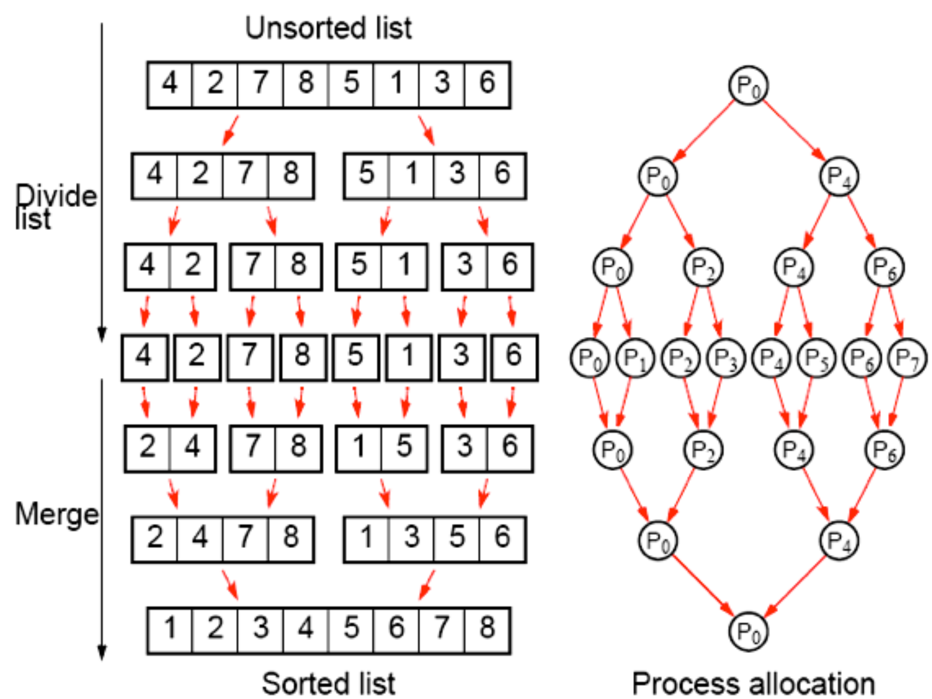
Parallel MergeSort

- Note: sorting two sub-arrays can be done in parallel. Therefore two recursive calls can be called in parallel.

- The first division phase is essentially scattering the array across the processors.

- The second merge phase can be done in parallel with each processor using a sequential merge operation.

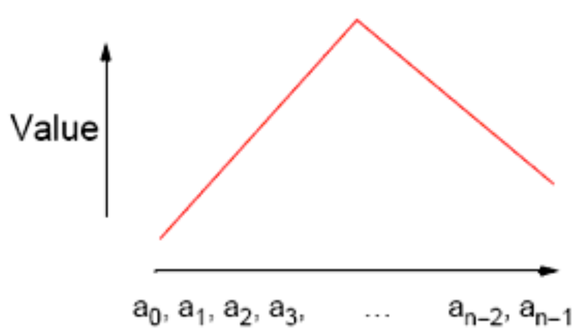
- The overall running time is O(n log n) / (log p) ~ O(n) but the unbalanced processor load and communication makes this algorithm inefficient than expected in practice.



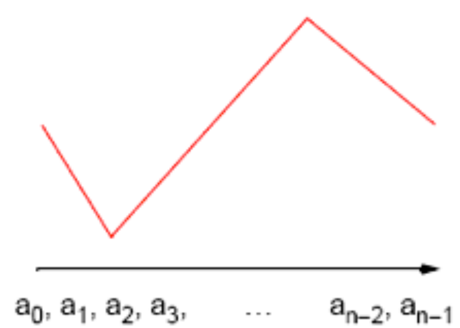
Bitonic MergeSort

- Bitonic Mergesort was introduced by K.E. Batcher in 1968.

- A monotonic sequence is a list that is increasing in value.
  - $a_0, a_1, a_2, \ldots a_{n-2}, a_{n-1}$ where $a_0 < a_1 < a_2 < \ldots a_{n-2} < a_{n-1}$
- A bitonic sequence is defined as a list with two sequences, one increasing and another decreasing; no more than one local minimum and one local maximum. (endpoints (i.e., wraparound) must be considered):
  - $a_0 < a_1 < a_2 < \ldots a_{i-1} < a_i > a_{i+1} \ldots > a_{n-2} > a_{n-1}$



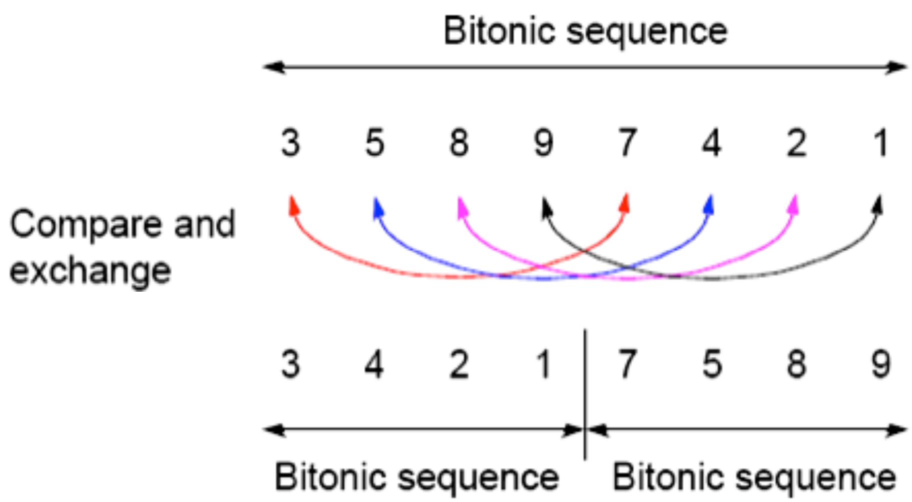
(a) Single maximum



(b) Single maximum and single minimum

Binary Split

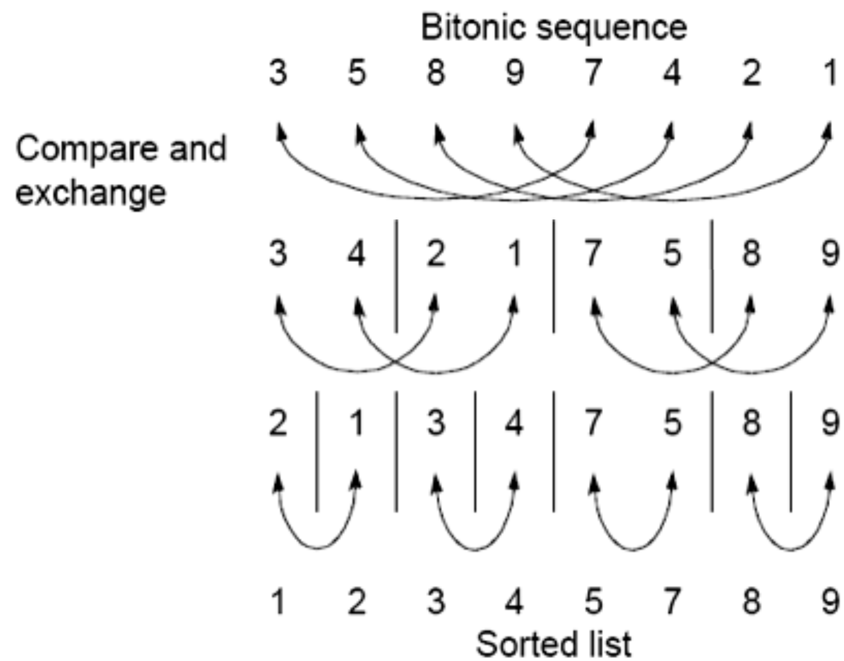
- Divide the bitonic list into two equal halves.
- Compare-Exchange each item on the first half (a_i) with the corresponding item in the second half ($a_{i+n/2}$).



- Result: Two bitonic sequences where the numbers in one sequence are all less than the numbers in the other sequence.

Sorting a Bitonic Sequence via Bitonic Splits

- Compare-and-exchange moves smaller numbers of each pair to left and larger numbers of pair to right.
- Given a bitonic sequence, recursively performing binary splits will sort the list.
- Q: How many binary splits does it takes to sort a list?
  A: **log n**

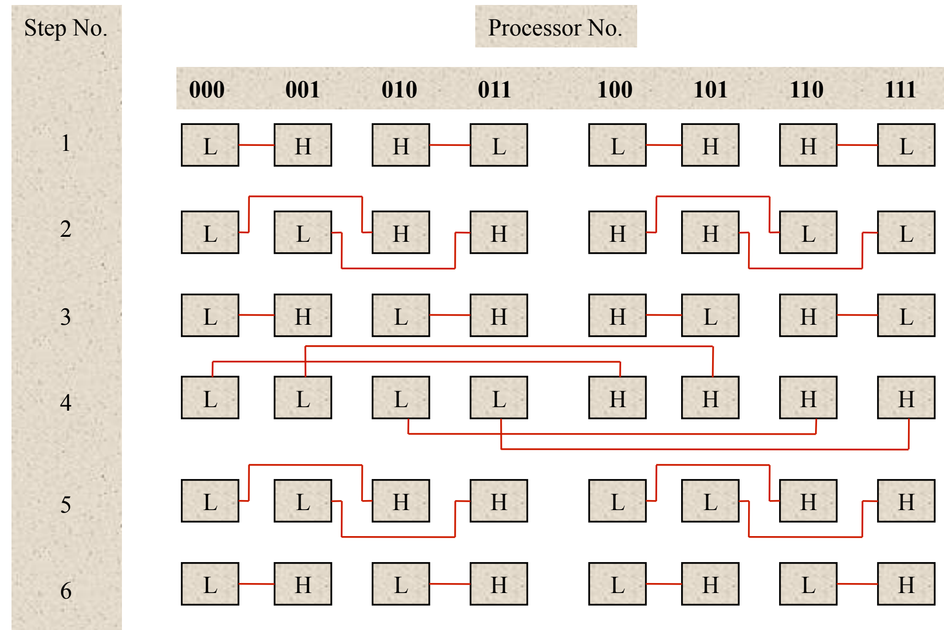


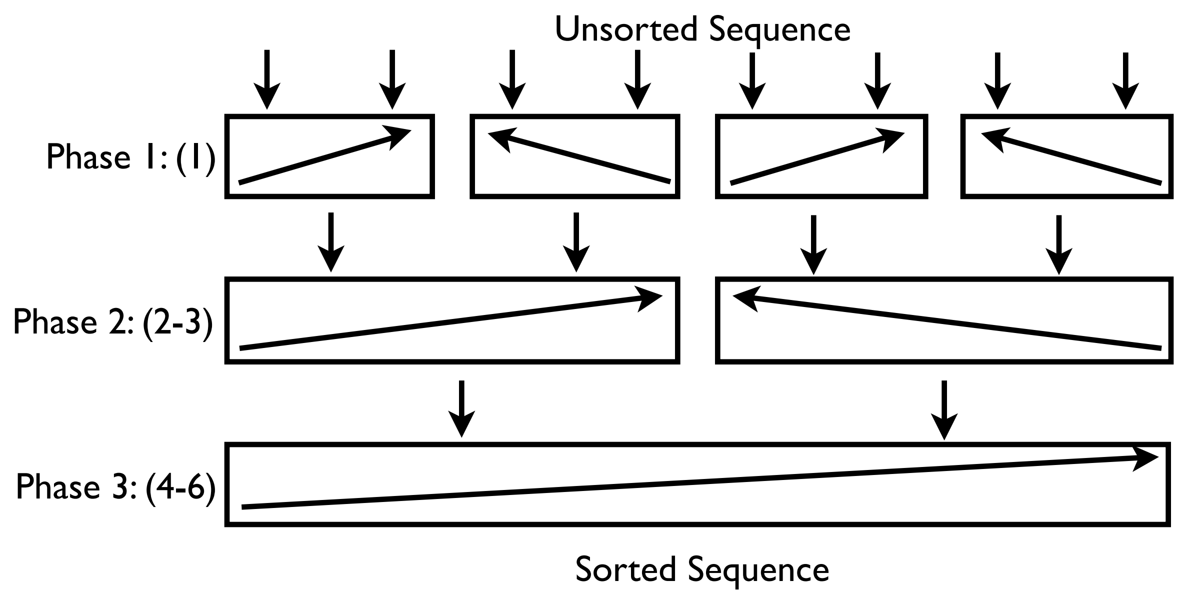
Sorting an Arbitrary Sequence via Bitonic Splits

- To sort an arbitrary sequence, A) generate a bitonic sequence, then B) sort it using a series of bitonic splits.

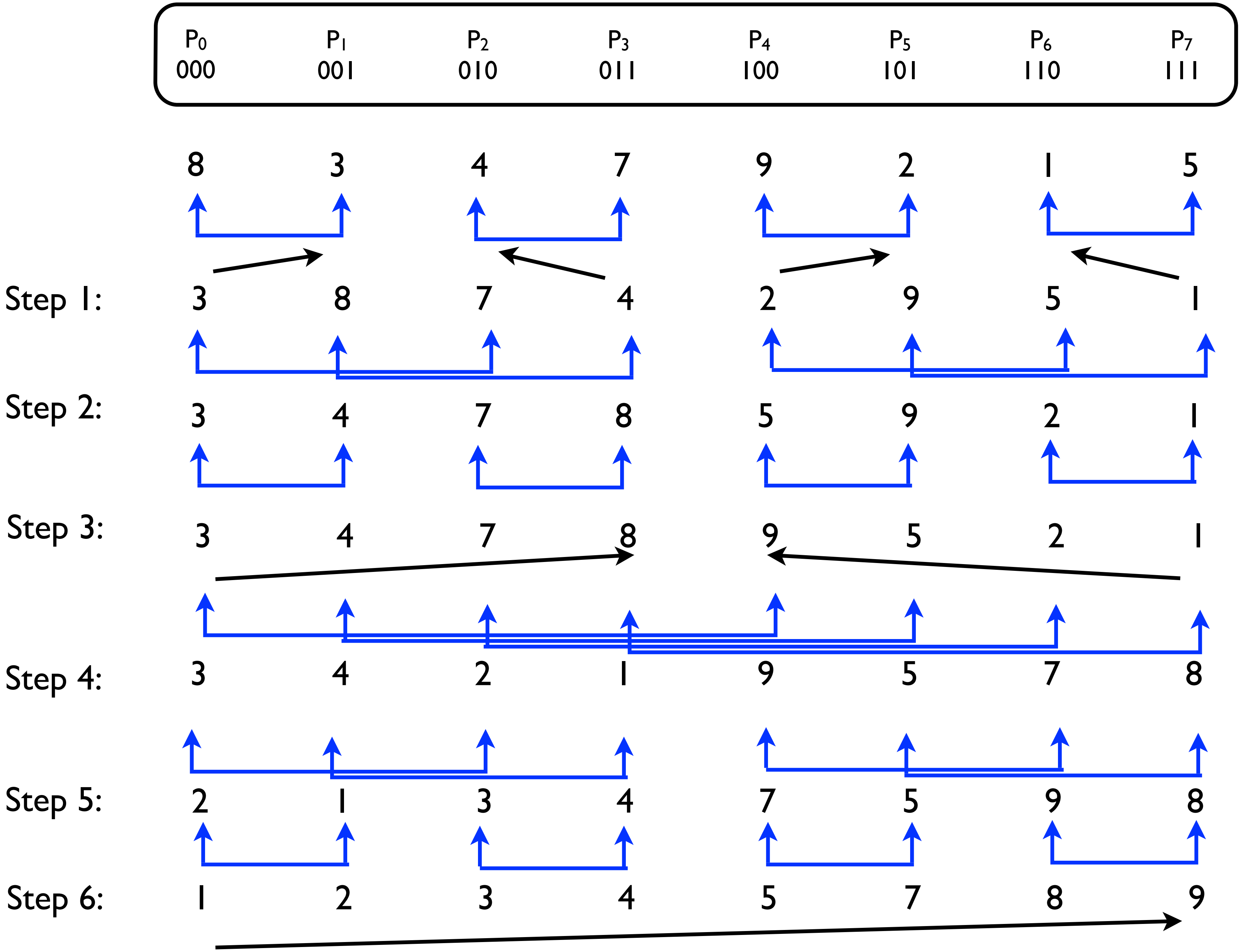
- To generate a bitonic sequence:
  - The unsorted sequences are merged into larger bitonic sequences, starting with pairs of adjacent numbers (Step 1).
  - By a compare-and-exchange operation, pairs of adjacent numbers formed into increasing sequences and decreasing sequences. Pairs form a bitonic sequence of twice the size of each original sequences. By repeating this process, bitonic sequences of larger and larger lengths obtained (Steps 2-3).

- Finally, a single bitonic sequence is sorted into a single increasing sequence (Steps 4-6).





Bitonic Sort Example



Bitonic Sort Analysis

- In order to form a sorted sequence of length n from two sorted sequences of length n/2, there are log(n) phases required (e.g. the 3 = log(8) phase to form a monotonic sequence i from two bitonic sequences j and j'). The number of phases T(n) of the entire sorting network is given by:
- T(n) = log(n) + T(n/2)
- The solution of this recurrence equation is:

$$T(n) = \sum_{i=1}^{k} k = \frac{k(k+1)}{2} = \frac{\log(n)(\log(n)+1)}{2}$$

- Therefore, the overall run time of the algorithm is $O(\log(n)^2)$.

Rank Sort

- Number of elements that are smaller than each selected element is counted. This count provides the position of the selected number, its "rank" in the sorted list.
- First a[0] is read and compared with each of the other numbers, a[1] … a[n-1], recording the number of elements less than a[0].
- Suppose this number is x. This is the index of a[0] in the final sorted list.
- The number a[0] is copied into the final sorted list b[0] … b[n-1], at location b[x]. Actions repeated with the other numbers.
- Overall sequential time complexity of rank sort: $T(n) = O(n^2)$

```
// Serial Rank Sort

for (i = 0; i < n; i++) { /* for each number */
   x = 0;
   for (j = 0; j < n; j++)
    /* count number less than it */
    if (a[i] > a[j])
       x++;
    /* copy number into correct place */
    b[x] = a[i];
}

// *This code needs to be fixed if
// duplicates exist in the sequence.
```

Parallel Rank Sort (P=n)

- One number is assigned to each processor.
- P_i finds the final index of a[i] in O(n) steps.

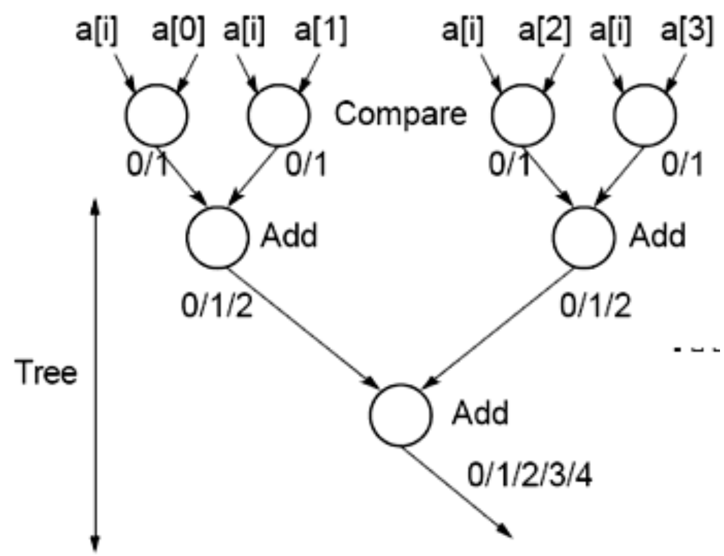
```
__global__ void ranksort (int* a, int *b, int n) {
  int i = blockIdx.x * blockDim.x + threadIdx.x

  if (i < n) {
    int x = 0;
    /* count number less than it */
    for (int j = 0; j < n; j++)
      if (a[i] > a[j])
        x++;
      /* copy no. into correct place */
      b[x] = a[i];
    }
  }
}
```

- Parallel time complexity, O(n), but that’s not all!

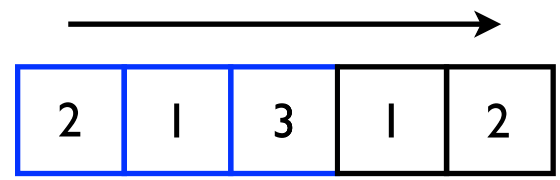
Parallel Rank Sort ($P=n^2$)

- Use n processors to find the rank of one element. The final count, i.e. rank of a[i] can be obtained using a global sum operation (e.g., reduction).
- Time complexity (for $P=n^2$): O(log n)

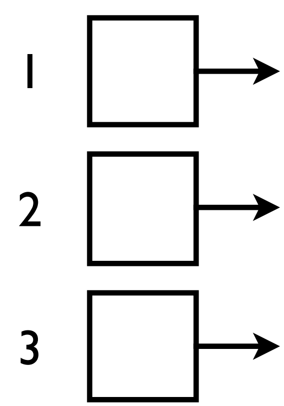


Bucket Sort

- For an array of N numbers, create M buckets (or bins) for the range of numbers in the array.



- Note in the example that there are two “2”s and two “1”s.



- Each of the elements are put into one of the M buckets.
- This is a stable sorting algorithm.

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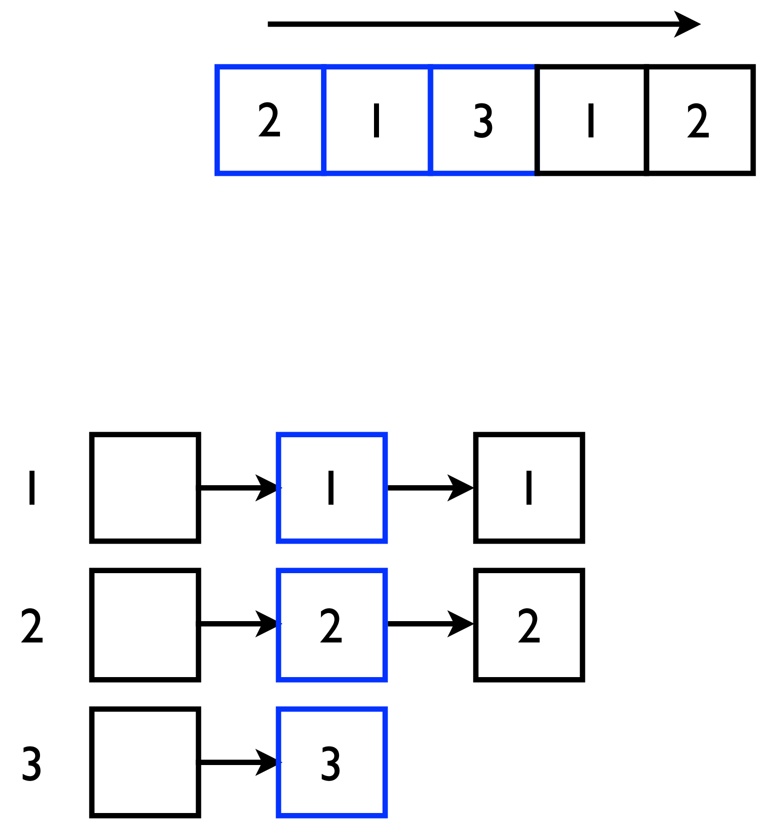

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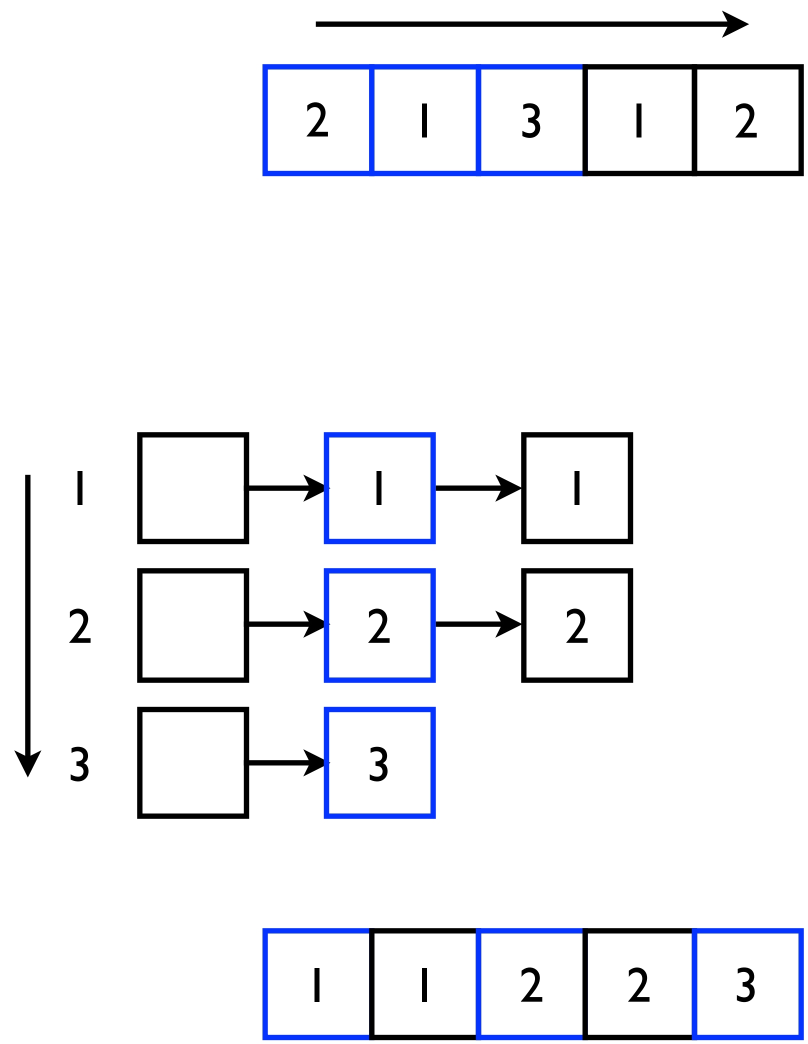
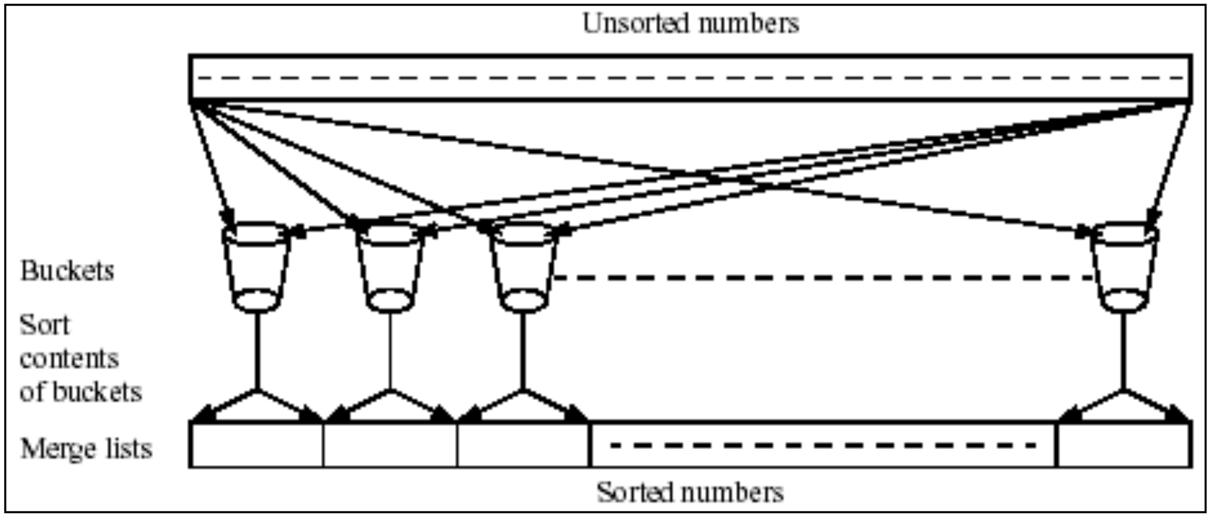


Bucket Sort



- Sequential sorting time complexity: O(n + m) for n numbers divided into m parts.
  - Placing into buckets is O(n).
  - Moving from buckets to sorted list is O(n + m).

- Works well if the original numbers uniformly distributed across a known interval, say 0 to a-1.

- Simple approach to parallelization: assign one processor for each bucket.

Radix Sort

- A radix is the number taken to be the base (or root) of a system of numbers. For example, for the binary system, the radix is 2, and for the decimal system, the radix is 10.

- Radix Sort is an integer sorting algorithm that uses bucket sort for each digit of an integer (keys) for a sequence of n integers starting from the least significant digit (LSD) to the most significant digit (MSD). The algorithm dates back to a patent in 1887 by Herman Hollerinth on tabulating machines.

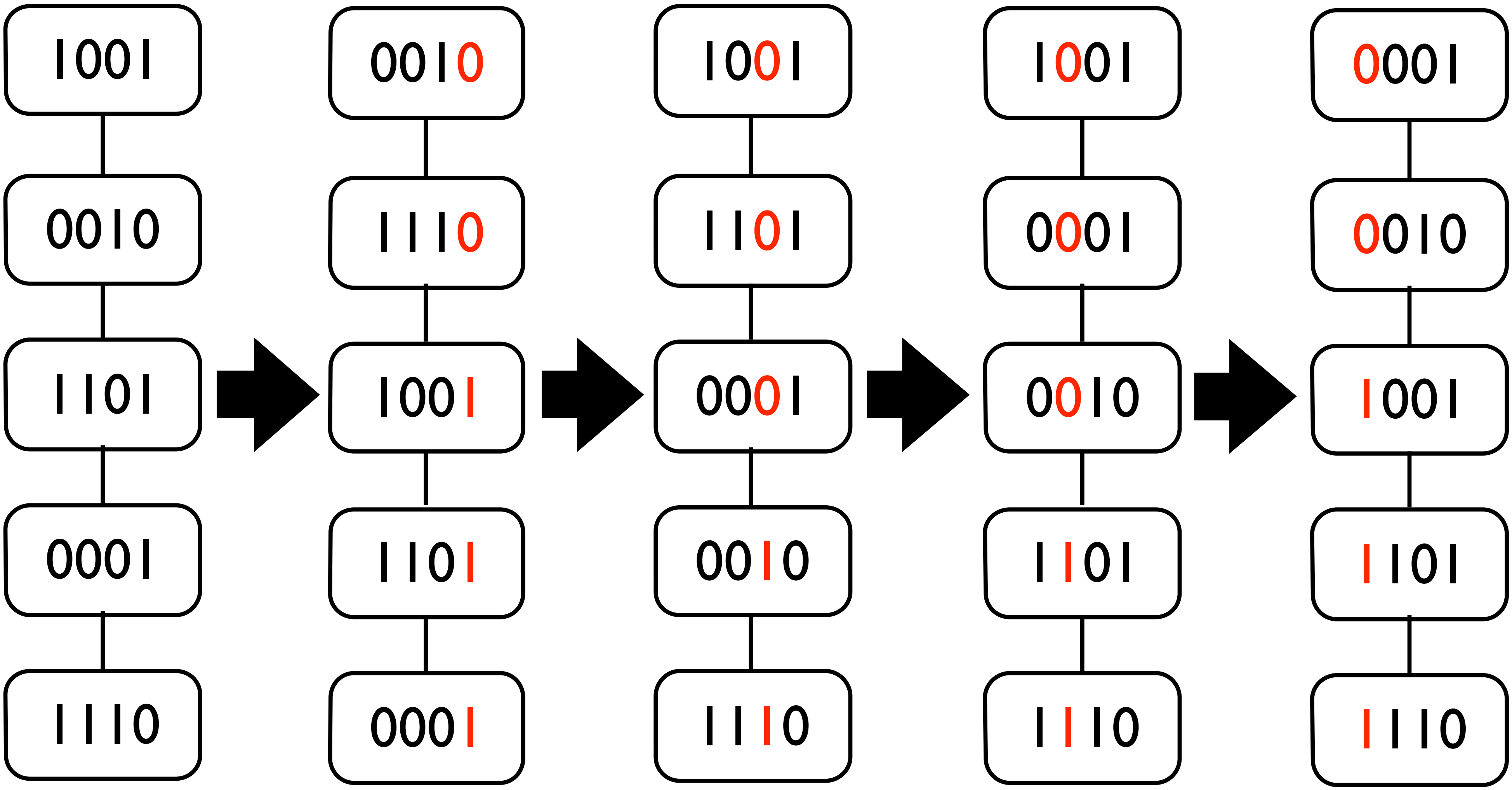
- Consider a sequence of n b-bit integers: $x = x_{b-1}...x_1x_0$

- For a set of binary numbers, we represent each element as a b-tuple of integers in the range [0,1] and apply radix sort with n=2.

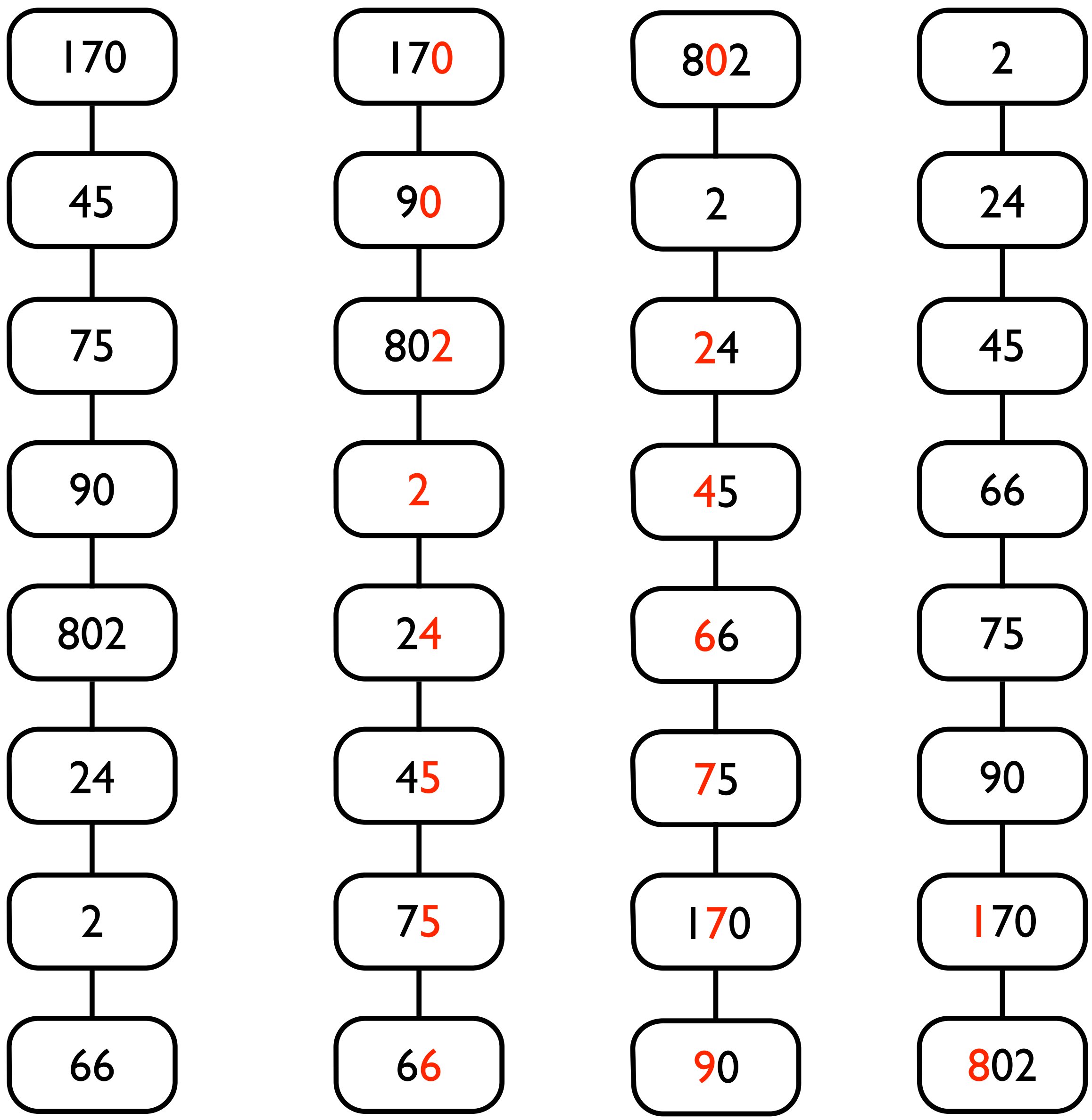
- Serial running time: $O(kn^2)$ where k is the number of digits.

- Parallel running time: $O(kn^2)/p \sim O(kn)$

Radix Sort Example 1



Radix Sort Example 2



Radix Sort Parallel Implementation

- Two approaches:
  - 1) Bucket sort each of the keys.
  - 2) Rank sort each of the keys.

Review

- Odd-Even Transposition Sort
- Merge Sort
- Bitonic Sort
- Rank Sort
- Bucket Sort
- Radix Sort